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Cosmological Entanglement

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Abstract We investigate the connection between the entanglement generated by expanding universe and the cosmological parameters. We show that the faster the universe expands and the larger the total volume results the higher degree of entanglement.

Keywords Bogoliubov transformation · Expanding universe · Von Neumann entropy

One source of the quantum corrections may be understood as entropy of entanglement, which arises when the density matrix of a pure quantum field theoretic state is reduced because the quantum field is not observed in some region of space.

Ball et al. [1] show that a dynamical spacetime generates entanglement between modes of a quantum field. Conversely, the entanglement encodes information concerning the underlying spacetime structure. Yasusada Nambu [2] investigate quantum entanglement of a scalar field in the inflationary universe. By introducing a bipartite system using a lattice model of scalar field. He apply the separability criterion based on the partial transpose operation and numerically calculate the bipartite entanglement between separate spatial regions. He find that the initial entangled state becomes separable or disentangled when the size of the spatial regions exceed the Hubble horizon. This is a necessary condition for the appearance of classicality of the quantum fluctuation.

Nambu et al. [3] investigated the behavior of the bipartite entanglement of the scalar field in the expanding universe, by introducing a bipartite system using a coarse-grained scalar field. They also investigate the condition of classicality that the quantum field can be treated as the classical stochastic variables.

Steeg et al. [4] show that entanglement can be used to detect spacetime curvature. This entanglement can be swapped to spatially separated quantum systems using standard local couplings. A single, inertial field detector in the exponentially expanding (de Sitter) vacuum responds as if it were bathed in thermal radiation in a Minkowski universe. They show that

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using two inertial detectors, interactions with the field in the thermal case will entangle certain detector pairs that would not become entangled in the corresponding de Sitter case.

Lin et al. [5] investigate the basic theoretical issues in the quantum entanglement of particle pairs created from the vacuum in a time-dependent background field or spacetime. Similar to entropy generation from these processes which depends on the choice of physical variables and how certain information is coarse-grained, entanglement dynamics hinges on the choice of measurable quantities and how the two parties are selected as well as the background dynamics of the field or spacetime.

In this letter we investigate the effects of cosmological parameters on entanglement generated between free modes of massive conformally coupled scalar field by expanding universe. We begin by presenting a brief review of scalar field in Robertson-Walker spacetime with line element

$$ds^2 = C(\eta)(d\eta^2 - dx^2 - dy^2 - dz^2), \tag{1}$$

where η is the conformal time. The Klein-Gordon equation in curved spacetime is

$$(\partial^\mu \partial_\mu + m^2 + \xi R)\varphi(x) = 0. \tag{2}$$

Here R is the scalar curvature, and ξ is a coupling constant. There are two popular choices for ξ: minimal coupling ($\xi = 0$) and conformal coupling ($\xi = 1/6$). The former leads to the simplest equation of motion, whereas the latter leads to a theory which is conformally invariant in four dimensions in the massless limit. The two bases will be related by Bogolubov coefficients in the usual way. Once we determine these coefficients we easily get the number of created particles per mode and from this the spectrum. For asymptotically flat spacetimes there are two different vacuum $|0\rangle_{in}$ and $|0\rangle_{out}$ associated with two Fock spaces $\mathcal{F}^{in}$ and $\mathcal{F}^{out}$. The *in* vacuum state is defined by $a_{in}(\mathbf{k})|0\rangle_{in} = 0$, $\forall \mathbf{k}$, and describes the situation when no particles are present initially. The out-vacuum state is defined by $a_{out}(\mathbf{k})|0\rangle_{out} = 0$, $\forall \mathbf{k}$ and describes the situation when no particles are present at late times. Assume that $\varphi_{\mathbf{k}}^{in}(x)$ and $\varphi_{\mathbf{k}}^{out}(x)$ are positive frequency solutions of conformally coupled Klein-Gordon equation in the remote past and far future. The *in* positive frequency mode can be expressed as a linear combination of the *out* positive and negative frequency modes

$$\varphi_{\mathbf{k}}^{in}(x) = \alpha_k \varphi_{\mathbf{k}}^{out}(x) + \beta_k \varphi_{-\mathbf{k}}^{*out}(x), \tag{3}$$

these relations and the complex coefficients α_k, β_k are called respectively the Bogoliubov transformation and the Bogoliubov coefficients. We investigate the entanglement entropy for this model of RW-spacetime. To see how, particle creation in a spacetime with Minkowskian *in* and *out* regions can create entanglement, we shall consider a simple example: doubly asymptotically static spacetime which is described by following scale factor [6]

$$C(\eta) = 1 + \epsilon(1 + \tanh \rho\eta), \tag{4}$$

with positive real parameters ϵ and ρ, controlling the total volume and rapidity of the expansion. In the distant past and far future, the spacetime becomes flat since $C(\eta)$ tends to $1 + 2\epsilon$ in $\eta \to +\infty$ or out-region and 1 as $\eta \to -\infty$ or in-region. In this model of spacetime *in* and *out* vacua are well defined because the scale factor reduces to a constant at asymptotic past and future times, therefore in these regions there is a timelike killing vector and one can defined particle state in terms of positive frequency modes.

After some algebra, the solutions of conformally coupled scalar field behaving as positive frequency modes as $\eta \to -\infty (t \to -\infty)$, are found to be

$$\varphi_{\mathbf{k}}^{in}(x) = (2\pi)^{-3/2}(2\omega_{in})^{-1/2} \exp\left(i\mathbf{k}\cdot\mathbf{x} - i\omega_+\eta - i\frac{\omega_-}{\rho}\ln[2\cosh\rho\eta] \right) \times F\left(1 + i\frac{\omega_-}{\rho}, i\frac{\omega_-}{\rho}; 1 - i\frac{\omega_{in}}{\rho}; \frac{1}{2}(1+\tanh\rho\eta) \right) \tag{5}$$

where

$$\omega_{in} = \sqrt{k^2 + m^2}, \tag{6}$$

$$\omega_{out} = \sqrt{\omega^2 + 2m^2\epsilon}, \tag{7}$$

$$\omega_{\pm} = \frac{1}{2}(\omega_{out} \pm \omega_{in}). \tag{8}$$

Now we are ready to calculate the Bogoliubov coefficients induced by the expanding universe. Using the linear transformation properties of hypergeometric functions at $z \longrightarrow 1$ [7]

$$F(a,b,c,z) \approx \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} + (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \tag{9}$$

Bogolubov coefficients are given by

$$\alpha_k = \sqrt{\frac{\omega_{out}}{\omega_{in}}} \frac{\Gamma\left(1 - i\frac{\omega_{in}}{\rho}\right)\Gamma\left(-i\frac{\omega_{out}}{\rho}\right)}{\Gamma\left(1 - \frac{i\omega_+}{\rho}\right)\Gamma\left(-\frac{i\omega_+}{\rho}\right)}, \tag{10}$$

$$\beta_k = \sqrt{\frac{\omega_{out}}{\omega_{in}}} \frac{\Gamma\left(1 - i\frac{\omega_{in}}{\rho}\right)\Gamma\left(i\frac{\omega_{out}}{\rho}\right)}{\Gamma(1 + \frac{i\omega_-}{\rho})\Gamma(\frac{i\omega_-}{\rho})}. \tag{11}$$

Using the following relations [7]

$$|\Gamma(iy)|^2 = \frac{\pi}{y\sinh\pi y}, \tag{12}$$

$$|\Gamma(1+iy)|^2 = \frac{\pi y}{\sinh\pi y}, \tag{13}$$

we get

$$|\alpha_k|^2 = \frac{\sinh^2\frac{\pi}{\rho}\omega_+}{\sinh\frac{\pi}{\rho}\omega_{in}\sinh\frac{\pi}{\rho}\omega_{out}}, \tag{14}$$

$$|\beta_k|^2 = \frac{\sinh^2\frac{\pi}{\rho}\omega_-}{\sinh\frac{\pi}{\rho}\omega_{in}\sinh\frac{\pi}{\rho}\omega_{out}}. \tag{15}$$

Quantum entanglement can be measured in many ways. For bipartite systems the von Neumann entropy is quite commonly used. Introducing the state vectors $|n,k\rangle_{in(out)}$ with occupation number n. It is possible to express the initial vacuum state in terms of the state vectors after the expansion of spacetime

$$|0\rangle_{in} = \sum_n C_n(k)|n, k\rangle_{out}|n, -k\rangle_{out}, \tag{16}$$

with

$$C_n(k) = \sqrt{1 - |\beta_k|^2/|\alpha_k|^2}(\beta_k/\alpha_k)^n. \tag{17}$$

Due to the existence of the non-zero coefficients $C_n(k)$, (16) show that there is some finite probability that the initial vacuum state will radiate pairs of particles moving in opposite directions. During expansion, universe remains homogeneous; therefore, space symmetry is preserved and the momentum of the field is conserved. Hence, for every particle emitted with momentum $-k$, another one will be emitted with momentum k. By tracing over the degrees of freedom $-k$ modes in the out region, we obtain the reduced density operator

$$\rho_k^{(0)} = \sum_{m=0}^{\infty} {}_{out}\langle m, -k|0\rangle_{in}\langle 0\rangle_{in}|m, -k\rangle_{out}. \tag{18}$$

In statistical mechanics, the von Neumann entropy is used to determine an equilibrium state: an equilibrium state of an isolated system is determined by maximizing the entropy. Thus, we expect that the entanglement entropy may be used to determine a quantum state. The von Neumann entropy is often seen as a generalization of the Shannon entropy from classical information theory. It is defined as

$$S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho). \tag{19}$$

It is most easily calculated from the nonzero eigenvalues λ_i of ρ_k as

$$S(\rho) = -\sum_i \lambda_i \log_2 \lambda_i. \tag{20}$$

The entanglement entropy associated with the vacuum state, arises when we trace over the fields in inaccessible regions. Then the von Neumann entropy of the reduced density matrix (18) is [1]

$$S_0 = \log_2\left(\frac{\gamma^{\frac{\gamma}{\gamma-1}}}{1-\gamma}\right), \tag{21}$$

where $\gamma = |\beta_k|^2/|\alpha_k|^2$.

Now we study the structure of the entanglement entropy for one-particle state. Consider the one particle state in the in-region

$$|1_k^{in}\rangle = a_k^{\dagger(in)}|0_{in},\rangle \tag{22}$$

using the Bogoliubov transformations we have

$$|1_k^{in}\rangle = (\alpha a_k^{\dagger(out)} - \beta a_{-k}^{(out)}) \sum_n C_n(k)|n, k\rangle_{out}|n, -k\rangle_{out}. \tag{23}$$

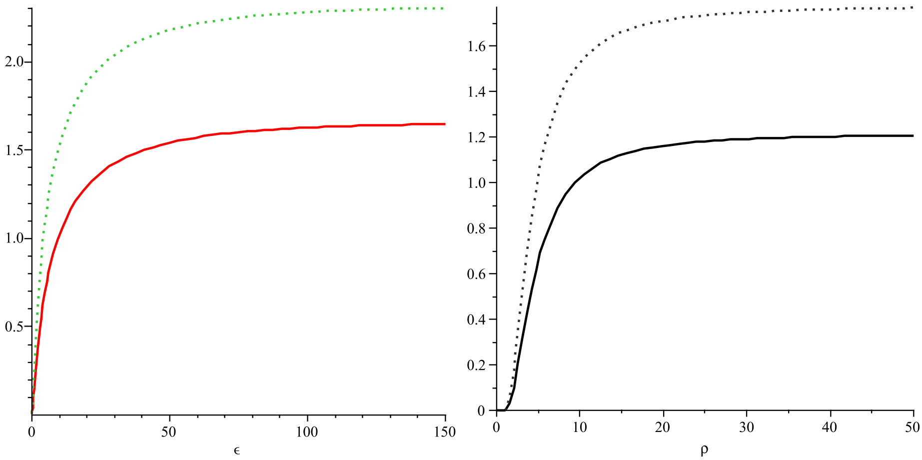

Fig. 1 Plot of entanglement entropy as a function of total volume of the expansion of the universe ϵ ($\rho = 10$) and the rapidity parameter of the expansion ρ ($\epsilon = 10$) for $k = m = 1$. *Dotted line* is S_1 and *solid line* is S_0

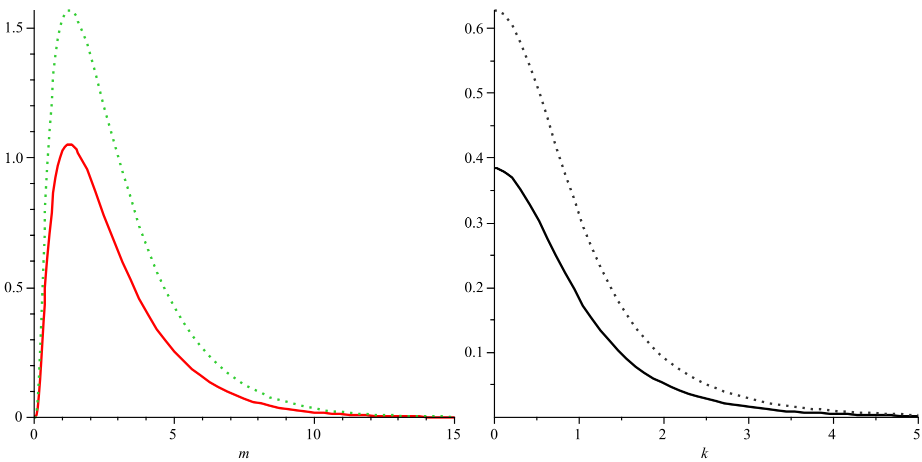

Fig. 2 Plot of entanglement entropy S versus m and k for $\rho = \epsilon = 10$. *Dotted line* is S_1 and *solid line* is S_0

By tracing over the degrees of freedom $-k$ modes in the out region, the reduced density matrix of (23) will be

$$\rho_k^{(1)} = (1-\gamma)^2 \sum_{n=1}^{\infty} n\gamma^{n-1} |n\rangle\langle n|. \tag{24}$$

The density operator is again diagonal then the eigenvalues of density matrix are

$$\lambda_n = (1-\gamma)^2 n\gamma^{n-1}. \tag{25}$$

Finally the entanglement entropy is readily calculated

$$S_1 = 2S_0 - (1-\gamma)^2 \sum_{n=1}^{\infty} n\gamma^{n-1} \log_2 n, \tag{26}$$

where S_0 is entanglement entropy of vacuum state in (21). In Fig. 1 we plot the entanglement entropy of vacuum and one-particle state as a function of ϵ, the total volume of the expansion of the universe and rapidity parameter of the expansion ρ. The faster the universe expands and the larger the total volume results the higher degree of entanglement. In Fig. 2 entanglement entropy is plotted as a function of mass and momentums of particles. The entanglement entropy of mass grows to a peak, declines again to near zero. The entanglement vanishes for sufficiently large values of mass and momentum.

References

1. Ball, J.L., Schuller, I.F., Schuller, F.P.: Phys. Lett. A **359**, 550 (2006)
2. Nambu, Y.: Phys. Rev. D **78**, 044023 (2008)
3. Nambu, Y., Ohsumi, Y.: Phys. Rev. D **80**, 124031 (2009)
4. Steeg, G.V., Menicucci, N.C.: Phys. Rev. D **79**, 044027 (2009)
5. Lin, S.-Y., Chou, C.-H., Hu, B.L.: Phys. Rev. D **81**, 084018 (2010)
6. Birrel, N.D., Davies, P.C.W.: Quantum Fields in Curved Space. Cambridge Univ. Press, Cambridge (1982)
7. Abramowitz, M., Stegun, I.: Handbook of Mathematical Functions. Dover, New York (1974)